## Problem 11376

Given a real number $a$ and a positive integer $n$, let

$$S_n(a) = \sum_{an<k\leq(a+1)n} \frac{1}{\sqrt{kn-an^2}}.$$

For which $a$ does the sequence $\langle S_n(a)\rangle$ converge?

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## Solution.

The sequence converges for any real value of $a$. In fact, as $n$ grows, the partial sum $S_n(a)$ converges to 2, as it will be explained.

The index bounds of the sum $S_n(a)$ are the real numbers $an$ and $(a+1)n$. If we conveniently choose an amount $\varepsilon$, such that $0<\varepsilon\leq 1$ and that $an+\varepsilon$ is the smallest integer larger than $an$, then the following numbers are integers as well:

$$\begin{gathered} an+\varepsilon+1 \\ an+\varepsilon+2 \\ \vdots \\ an+\varepsilon+n-1 \end{gathered}$$

By construction, we also have that the greatest integer smaller than $(a+1)n$ is $(a+1)n+\varepsilon-1$.

With this consideration, we can write the partial sum with integer index bounds:

$$\begin{aligned} S_n(a) = \sum_{an<k\leq(a+1)n} \frac{1}{\sqrt{kn-an^2}} &= \sum_{k=an+\varepsilon}^{an+\varepsilon+n-1} \frac{1}{\sqrt{kn-an^2}} \\ &= \sum_{k=0}^{n-1} \frac{1}{\sqrt{(k+an+\varepsilon)n-an^2}} \\ &= \sum_{k=0}^{n-1} \frac{1}{\sqrt{(k+\varepsilon)n}} \\ &= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \frac{1}{\sqrt{k+\varepsilon}} \end{aligned}$$

Hence the sequence $\langle S_n(a)\rangle$ converges if and only if the following limit exists:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \frac{1}{\sqrt{k+\varepsilon}} = L \tag{1}$$

We consider the following theorem:

**Theorem:** Let $f(x)$ be a decreasing continuous function in the closed interval $[a-1,b+1]$, with $a$ and $b$ integers. Left and right Riemann sums yield:

$$\int_a^{b+1} f(x)dx \leq \sum_{k=a}^{b} f(k) \leq \int_a^{b+1} f(x-1)dx$$

This is not immediatly applicable to the partial sum $S_n(a)$, for the inner function has one singularity at $x=-\varepsilon$. Since the inferior limit of the sum is $0$, the hypotesis is not satisfied, so the right inequality cannot be stated. We fix this by splitting the sum:

$$\begin{aligned} \int_0^n \frac{1}{\sqrt{x+\varepsilon}}dx &\leq \sum_{k=0}^{n-1} \frac{1}{\sqrt{k+\varepsilon}} \\ &= \frac{1}{\sqrt{\varepsilon}} + \sum_{k=1}^{n-1} \frac{1}{\sqrt{k+\varepsilon}} \leq \frac{1}{\sqrt{\varepsilon}} + \int_1^n \frac{1}{\sqrt{x-1+\varepsilon}}dx \end{aligned}$$

Therefore L is nested by:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\int_0^n \frac{1}{\sqrt{x+\varepsilon}}dx\right) \le L \le \lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{\varepsilon}}+\int_1^n \frac{1}{\sqrt{x-1+\varepsilon}}dx\right)$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(2\sqrt{x+\varepsilon}\right)\Big|_0^n \le L \le \lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{\varepsilon}}+\frac{1}{\sqrt{n}}\left(2\sqrt{x-1+\varepsilon}\right)\Big|_1^n$$

Considering that $0 < \varepsilon \le 1$:

$$2 \le L \le 2$$

Which yields as only possibility:

$$L = \lim_{n\to\infty} S_n(a) = 2$$

Note that the subradical quantity in each term is positive, since $kn > (an)n = an^2$. This proves that the sequence converges to 2, for any real value of "$a$".

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We provide a generalization of the problem. For a real number $p$, let

$$S_n(a) = \sum_{an<k\le(a+1)n} \frac{1}{(kn-an^2)^p}.$$

With the same procedure given before, the sequence $\langle S_n(a)\rangle$ converges to

$$L = \lim_{n\to\infty}\frac{1}{n^p}\left(\int_0^n \frac{dx}{(x+\varepsilon)^p}\right)$$

- $L = \infty$ if $p < 1/2$.
- $L = 2$ if $p = 1/2$.
- $L = 0$ if $p > 1/2$.

Of course, the last interval includes the value $p = 1$, for which, by L'Hopital's rule:

$$L = \lim_{n\to\infty}\frac{\ln(n+\varepsilon)-\ln(\varepsilon)}{n^p} = 0.$$

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