A walk through Determinants and Recurrence Relations

11396. *Proposed by Gérard Letac, Université Paul Sabatier, Toulouse, France.* For complex z, let $H_n(z)$ denote the $n \times n$ Hermitian matrix whose diagonal elements all equal 1 and whose above-diagonal elements all equal z. For $n \geq 2$, find all z such that $H_n(z)$ is positive semi-definite.

Solution by Francisco Vial (student), Pontificia Universidad de Chile, Santiago de Chile. We will use Dodgson's rule for determinants to find an explicit formula for $\det(H_n(z))$:

$$\det\left[(a_{i,j})_{\substack{1\leq i\leq n\\1\leq j\leq n}}\right]\cdot\det\left[(a_{i,j})_{\substack{2\leq i\leq n-1\\2\leq j\leq n-1}}\right] =$$

$$\det\left[(a_{i,j})_{\substack{1\leq i\leq n-1\\1\leq j\leq n-1}}\right]\cdot\det\left[(a_{i,j})_{\substack{2\leq i\leq n\\2\leq j\leq n}}\right] - \det\left[(a_{i,j})_{\substack{1\leq i\leq n-1\\2\leq j\leq n}}\right]\cdot\det\left[(a_{i,j})_{\substack{2\leq i\leq n\\1\leq j\leq n-1}}\right].$$

Here, $\det(M) \equiv |M|$ if M is any $n \times n$ matrix. $H_n(z)$ is defined as

$$H_n(z) = \begin{pmatrix} 1 & z & z & \dots & z \\ z^* & 1 & z & \dots & z \\ z^* & z^* & 1 & \dots & z \\ \vdots & \vdots & \vdots & \dots & \vdots \\ z^* & z^* & z^* & \dots & 1 \end{pmatrix}_{n\times n}$$

and it is easy to see that

$$\det\left[(h_{i,j})_{\substack{2\leq i\leq n-1\\2\leq j\leq n-1}}\right] = |H_{n-2}|,$$

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Hence, by Dodgson's rule,

$$|H_n|\cdot|H_{n-2}| = |H_{n-1}|^2 - |A_{n-1}|\,|B_{n-1}|, \tag{1}$$

where

$$A_n(z) = \begin{pmatrix} z & z & z & \dots & z \\ 1 & z & z & \dots & z \\ z^* & 1 & z & \dots & z \\ \vdots & \vdots & \vdots & \dots & \vdots \\ z^* & z^* & z^* & \dots & z \end{pmatrix}_{n\times n}, \quad B_n(z) = \begin{pmatrix} z^* & 1 & z & \dots & z \\ z^* & z^* & 1 & \dots & z \\ z^* & z^* & z^* & \dots & z \\ \vdots & \vdots & \vdots & \dots & \vdots \\ z^* & z^* & z^* & \dots & z^* \end{pmatrix}_{n\times n}.$$

We can compute $|A_n(z)|, |B_n(z)|$ in terms of $|H_n(z)|$ and $|H_{n-1}(z)|$ as follows:

$$\begin{aligned}\det(A_n) &= \begin{vmatrix} z & z & z & \dots & z \\ 1 & z & z & \dots & z \\ z^* & 1 & z & \dots & z \\ \vdots & \vdots & \vdots & \dots & \vdots \\ z^* & z^* & z^* & \dots & z \end{vmatrix} = (-1)^{n-1}\begin{vmatrix} 1 & z & z & \dots & z \\ z^* & 1 & z & \dots & z \\ z^* & z^* & 1 & \dots & z \\ \vdots & \vdots & \vdots & & \\ z & z & z & \dots & z \end{vmatrix} \\ &= (-1)^{n-1}\left(\frac{z}{z^*}\right)\begin{vmatrix} 1 & z & z & \dots & z \\ z^* & 1 & z & \dots & z \\ z^* & z^* & 1 & \dots & z \\ \vdots & \vdots & \vdots & & \\ z^* & z^* & z^* & \dots & z^* \end{vmatrix} \\ &= (-1)^{n-1}\left(\frac{z}{z^*}\right)\left((z^*-1)\,|H_{n-1}| + |H_n|\right).\end{aligned}$$

(Note that $(n-1)$ rows have been permuted, then the last row has been multiplied by $\frac{z^*}{z}$ and finally the determinant has been computed using Laplace expansion on the last column).

In a similar fashion we have

$$\det(B_n) = (-1)^{n-1}\left(\frac{z^*}{z}\right)\left((z-1)\,|H_{n-1}| + |H_n|\right).$$

Replacing in (1) we obtain

$$\begin{aligned} |H_n|\cdot|H_{n-2}| &= |H_{n-1}|^2 - \left((z^*-1)\,|H_{n-2}| + |H_{n-1}|\right)\left((z-1)\,|H_{n-2}| + |H_{n-1}|\right) \\ &= (2-z-z^*)\,|H_{n-1}|\,|H_{n-2}| - (z-1)(z^*-1)\,|H_{n-2}|^2 . \end{aligned}$$

We now have a linear recurrence relation for $|H_n|$:

$$|H_n| = (2-z-z^*)\,|H_{n-1}| - (z-1)(z^*-1)\,|H_{n-2}|.$$

It is easy to see that the roots of the characteristic equation

$$r^2 - (2-z-z^*)r + (z-1)(z^*-1) = 0$$

are

$$\begin{aligned} r_1 &= 1-z\ , \\ r_2 &= 1-z^*\ . \end{aligned}$$

Hence,

$$\left\{\begin{array}{lcc} |H_n| &=& \alpha r_1^n + \beta r_2^n \\ |H_1| &=& 1 \\ |H_2| &=& 1-zz^*. \end{array}\right.$$

Solving for α, β yields

$$\begin{aligned} |H_n| &= \frac{z^*}{z^*-z}(1-z)^n + \frac{z}{z-z^*}(1-z^*)^n \\ |H_n| &= 2\Re\left(\frac{z}{z-z^*}(1-z^*)^n\right). \end{aligned}$$

If we write $1-z=\rho e^{i\theta}$, $1-z^*=\rho e^{-i\theta}$, with $\rho\geq 0$, $-\pi\leq\theta<\pi$, $i.e.$ polar coordinates with the origin at $z=1$, $|H_n(z)|$ takes the form:

$$|H_n| = \frac{\rho^{n-1}}{\sin\theta}\left(\rho\sin\left((n-1)\theta\right) - \sin(n\theta)\right).$$

Therefore, if $z=1-\rho e^{i\theta}$, H_n is positive semi-definite if and only if each subdeterminant is non negative, $i.e.$,

$$\Re\left(\frac{z}{z-z^*}(1-z^*)^k\right) = \frac{1}{\sin\theta}\left(\rho\sin\left((k-1)\theta\right) - \sin(k\theta)\right) \geq 0 \ \ , \ \ k=2,3,\ldots,n.$$

We proceed to analize the case $z=z^*\equiv a$, which of course is not included in the last condition for z. The characteristic equation takes the form

$$r^2 - 2(1-a)r + (1-a)^2 = 0 \ \ \Rightarrow\ \ r = 1-a,$$

hence,

$$\left\{\begin{array}{lcc} |H_n| &=& (1-a)^n(\alpha+\beta n) \\ |H_1| &=& 1 \\ |H_2| &=& 1-a^2. \end{array}\right.$$

Solving for α, β yields

$$|H_n(a)| = (1-a)^n + a(1-a)^{n-1}n$$

and we can show with elementary algebra that the inequalities $|H_k|\geq 0\ \ k=1,2,\ldots,n$ reduce to

$$\frac{-1}{n-1} \leq \frac{-1}{k-1} \leq a < 1\ ,\ \ k=2,3,\ldots,n$$

$$\Rightarrow \frac{-1}{n-1} \leq a \leq 1$$

(The case $a=1$ is part of the solution, for $|H_n(1)|=0\ \ \forall\, n\geq 2$.)

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