**A product.**

**906.** *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH.*

(a) Find the value of

$$\prod_{n=1}^{\infty}\left(\frac{2m+2n-1}{2n-1}\right)\left(\frac{2n}{2m+2n}\right),$$

where $m$ denotes a positive integer.

(b) More generally, if the real part of $z$ is positive, find the value of

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{(-1)^{n-1}}.$$

*Solution by Francisco Vial (student), MT group, Pontificia Universidad Católica de Chile, Santiago, Chile.*

(a) Consider the partial products

$$P_N := \prod_{n=1}^{N}\left(\frac{2m+2n-1}{2n-1}\right)\left(\frac{2n}{2m+2n}\right)$$

and let $P=\lim P_N$ be the product given. For $N>m$, $P_N$ telescopes as follows

$$\begin{aligned}
\prod_{n=1}^{N}\left(\frac{2m+2n-1}{2n-1}\right)\left(\frac{2n}{2m+2n}\right) &= \prod_{n=1}^{N}\left(\frac{2m+2n-1}{2m+2n}\right)\left(\frac{2n}{2n-1}\right)\\
&= \frac{\prod_{n=1}^{N}\left(1-\frac{1}{2m+2n}\right)}{\prod_{n=1}^{N}\left(1-\frac{1}{2n}\right)}\\
= \frac{\prod_{n=m+1}^{N+m}\left(1-\frac{1}{2n}\right)}{\prod_{n=1}^{N}\left(1-\frac{1}{2n}\right)} &= \frac{\prod_{n=N+1}^{N+m}\left(1-\frac{1}{2n}\right)}{\prod_{n=1}^{m}\left(1-\frac{1}{2n}\right)}\\
&= \frac{\prod_{n=1}^{m}\left(1-\frac{1}{2n+2N}\right)}{\prod_{n=1}^{m}\left(1-\frac{1}{2n}\right)}
\end{aligned}$$

Therefore,

$$P=\lim_{N\to\infty}P_N=\prod_{n=1}^{m}\left(1-\frac{1}{2n}\right)^{-1},$$

since each of the finite terms of the product of the numerator tends to 1. This allows to conclude

$$P=\prod_{n=1}^{m}\left(\frac{2n}{2n-1}\right)=\frac{(2m)!!}{(2m-1)!!}=\frac{m!\sqrt{\pi}}{\Gamma(m+\frac{1}{2})}.$$

where $a!!$ denotes the double factorial of $a$.

(b) Write

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{(-1)^{n-1}} = \prod_{n=1}^{\infty}\left(1+\frac{z}{2n-1}\right)\left(1+\frac{z}{2n}\right)^{-1}$$
$$= \prod_{n=1}^{\infty}\left(\frac{z+2n-1}{2n-1}\right)\left(\frac{2n}{z+2n}\right)$$

and the same procedure given above works (or simply take $2m = z$) and one concludes

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{(-1)^{n-1}} = \frac{\Gamma(\frac{z}{2}+1)\sqrt{\pi}}{\Gamma(\frac{z+1}{2})}.$$